An Adaptive Deflation Domain-Decomposition Preconditioner for Fast Frequency Sweeps

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Adaptive multi-point model-order reduction is a well established methodology for computing fast frequency sweeps of finite-element models. However, in the case of electrically large structures, generating the reduced-order system is computationally expensive, because both the size of the finite-element model and the number of expansion points become large. Thus, a great number of independent large-scale systems of linear equations must be solved by iterative methods. To alleviate this problem, the present paper proposes to employ the reduced-order system already available at a given adaptive step for constructing an efficient two-level preconditioner. A numerical example demonstrates the benefits of the suggested approach.

Index Terms—**Reduced-order systems, finite-element analysis, iterative methods, phased arrays.**

I. Introduction

The FINITE-ELEMENT (FE) method is commonly used for simulating electromagnetic structures in the frequency domain. However, the solution process becomes expensive when the model domain is electrically large. In addition to the large volume to be discretized, extra h refinement or p enrichment becomes necessary to counter numerical dispersion, which increases the dimension of the FE system even more. When direct solvers are no longer applicable because of their high computational complexity, one must resort to iterative techniques. For these, the availability of efficient preconditioners is of utmost importance. For electrically large domains, non-overlapping domain decomposition (DD) methods are very appealing [1]–[3]. When a single frequency does not suffice but broadband analysis is required, multi-point methods of model-order reduction (MOR) are much more powerful than conventional FE analysis. The general idea is to generate a low-dimensional approximation space from well-chosen snapshots of the solution manifold. State-of-the-art MOR approaches place expansion points adaptively, based on some a posteriori error indicator [4]. In the case of electrically large structures, constructing the MOR basis is computationally expensive, because the number of expansion points tends to be large, and each of them requires one iterative solution of the large-scale FE system. We here propose to add an extra correction step inside the Krylov iteration. In contrast to Krylov recycling [5], it is based on the MOR basis available at the respective adaptive step. Thus the quality of the preconditioner and, in consequence, the number of Krylov iterations improve greatly during the process of MOR generation.

II. Finite-Element-Domain-Decomposition Formulation

We write E for the electric field strength, k_0 for the free-space wavenumber, μ_r for the relative magnetic permeability, γ_Γ for the tangential trace map, and π_Γ for the tangential component trace map, respectively. We consider a domain $\Omega \subset \mathbb{R}^3$ and, for simplicity, its decomposition into $N_s = 2$ non-overlapping subdomains such that $\Omega = \Omega_1 \bigcup \Omega_2$, $\Omega_1 \bigcap \Omega_2 = \emptyset$. To render the decomposed boundary value problem (BVP) equivalent to the original one, additional transmission-conditions (TCs) on the interface $\Gamma_{12} = \partial\Omega_1 \bigcap \partial\Omega_2$ are required. We here consider a TC with two transverse derivatives of second order [2],

$$\begin{aligned}\Gamma(E_i) = {} & \gamma_\Gamma(\mu_{ri}^{-1}\nabla \times E_i) + \alpha\pi_\Gamma(E_i) + \beta\nabla_\Gamma \times \nabla_\Gamma \times \pi_\Gamma(E_i) \\ & + \gamma\nabla_\Gamma\nabla_\Gamma \cdot \gamma_\Gamma(\mu_{ri}^{-1}\nabla \times E_i) \qquad \text{for } i \in \{1,2\},\end{aligned} \tag{1}$$

wherein $\alpha, \beta, \gamma \in \mathbb{C}$ are frequency- and material-dependent parameters. The resulting FE-DD system [2] is of the form

$$\begin{pmatrix} \mathbf{A}_1(k_0,\xi) & \mathbf{C}_{12}(k_0,\xi) \\ \mathbf{C}_{21}(k_0,\xi) & \mathbf{A}_2(k_0,\xi) \end{pmatrix} \begin{pmatrix} \mathbf{x}_1(k_0) \\ \mathbf{x}_2(k_0) \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1(k_0) \\ \mathbf{b}_2(k_0) \end{pmatrix}, \tag{2}$$

with $\xi = (\alpha,\beta,\gamma), \mathbf{A}_i \in \mathbb{C}^{N_i \times N_i}$, and $\mathbf{C}_{ij} \in \mathbb{C}^{N_i \times N_j}$, with $i, j \in \{1,2\}$. The vectors $\mathbf{x}_i$ contain N_i FE degrees of freedom for the electric field inside the sub-domain Ω_i as well as the auxiliary variables on the interface Γ_{12}.

III. Reduced-Order Model

As a prerequisite for projection-based MOR, the FE-DD system (2) is rewritten in affinely parametrized form with respect to k_0,

$$\Big(\sum_{i=1}^{I} \phi_i(k_0)\hat{\mathbf{A}}_i\Big)\hat{\mathbf{x}}(k_0) = \Big(\sum_{j=1}^{J} \theta_j(k_0)\hat{\mathbf{b}}_j\Big), \tag{3}$$

$$\mathbf{y}(k_0) = \Big(\sum_{j=1}^{J} \eta_j(k_0)\hat{\mathbf{b}}_j^T\Big)\hat{\mathbf{x}}(k_0), \tag{4}$$

with wavenumber-dependent functions $\phi_i, \theta_j, \eta_j : \mathbb{R} \to \mathbb{C}$ and block matrices and vectors defined as

$$\hat{\mathbf{A}}_i = \begin{bmatrix} \mathbf{A}_{1,i} & \mathbf{C}_{12,i} \\ \mathbf{C}_{21,i} & \mathbf{A}_{2,i} \end{bmatrix} \in \mathbb{C}^{(N_1+N_2)\times(N_1+N_2)}, \tag{5a}$$

$$\hat{\mathbf{x}}(k_0) = \begin{bmatrix} \mathbf{x}_1(k_0) \\ \mathbf{x}_2(k_0) \end{bmatrix} \in \mathbb{C}^{(N_1+N_2)}, \tag{5b}$$

$$\hat{\mathbf{b}}_i = \begin{bmatrix} \mathbf{b}_{1,i} \\ \mathbf{b}_{2,i} \end{bmatrix} \in \mathbb{C}^{(N_1+N_2)}. \tag{5c}$$

Table I: Computational Data*

Model	ROM	FE-DD	FE-DD
Dimension	16	$8 \cdot 10^6$	$8 \cdot 10^6$
Type of preconditioner	-	$\mathbf{M}^{-1}$	$\mathbf{P}_{\text{AD}}$
ROM construction (h) -	-	6.3	2.5
ROM evaluation (s)	$5 \cdot 10^{-3}$	-	-

* Matlab prototype code on Intel Core i5-4570 CPU @ 3.2 GHz.

The multi-point reduced-order model (ROM) is built from the FE solutions $\mathbf{x}(k_0^i)$ of (3) at the sampling points $k_0^i, \ldots, k_0^M$, selected adaptively by a greedy sampling procedure [4]. Thus, ROM construction requires the solution of the large-scale FE-DD system (3) at each parameter point. Since (3) is solved iteratively, ROM construction time is determined mainly by the convergence behavior of the linear solver.

To improve solver convergence, we propose to add an extra correction step inside the Krylov iteration: Let V_A denote the approximation space spanned by the snapshots $\mathbf{x}(k_0^i)$, $i = 1 \ldots A$. The $(A{+}1)$-th basis vector $\mathbf{x}(k_0^{A+1})$ is calculated with the help of an adaptive two-level preconditioner [6], given by

$$\mathbf{P}_{\text{AD}} = \mathbf{M}^{-1}\left(\mathbf{I} - \mathbf{A}\mathbf{V}_A\mathbf{E}^{-1}\mathbf{V}_A^*\right) + \mathbf{V}_A\mathbf{E}^{-1}\mathbf{V}_A^*, \quad 1 \le A \le M, \tag{6}$$

$$\mathbf{E} = \mathbf{V}_A^*\mathbf{A}\mathbf{V}_A, \tag{7}$$

$$\operatorname{range}(\mathbf{V}_A) = \operatorname{span}\left\{\mathbf{x}(k_0^i)\right\}, \qquad i = 1 \ldots A, \qquad A \le M, \tag{8}$$

wherein $\mathbf{V}_A\mathbf{E}^{-1}\mathbf{V}_A^T$ acts as a coarse-space correction. The operator $\mathbf{M}^{-1}$ describes a block Gauss-Seidel preconditioner containing the lower triangular part of the system matrix (2). After computing the projection space V^M given by

$$V^M = \operatorname{span}\left\{\mathbf{x}(k_0^i)\right\}, \qquad i = 1 \ldots M, \qquad M \ll N_1 + N_2, \tag{9}$$

Galerkin projection leads to a ROM of the form

$$\left(\sum_{i=1}^{I} \phi_i(k_0)\tilde{\mathbf{A}}_i\right)\tilde{\mathbf{x}}(k_0) = \sum_{j=1}^{J} \theta_j(k_0)\tilde{\mathbf{b}}_j, \tag{10}$$

$$\tilde{\mathbf{y}}(k_0) = \left(\sum_{j=1}^{J} \eta_j(k_0)\tilde{\mathbf{b}}_j^T\right)\tilde{\mathbf{x}}(k_0), \tag{11}$$

wherein the reduced matrices and vectors are defined as

$$\tilde{\mathbf{A}}_i = \mathbf{V}_M^*\hat{\mathbf{A}}_i\mathbf{V}_M \in \mathbb{C}^{M\times M}, \tag{12}$$

$$\tilde{\mathbf{b}}_i = \mathbf{V}_M^*\hat{\mathbf{b}}_i \in \mathbb{C}^{M}, \tag{13}$$

$$\operatorname{range}(\mathbf{V}_M) = V^M, \qquad \mathbf{V} \in \mathbb{C}^{N\times M}. \tag{14}$$

IV. Numerical Example

We consider an array consisting of 20×20 patch antennas in the frequency band $f \in [8, 12]$ GHz. The geometry of a single element is shown in Fig. 1. The FE system (3) is solved by the restarted GMRES(30) iterative method with stopping criterion $\delta = 10^{-6}$. The termination criterion for the ROM is $\sigma = 10^{-6}$. Table I gives computational data: Compared to the standard one-level preconditioner $\mathbf{M}^{-1}$, the proposed two-level preconditioner $\mathbf{P}_{\text{AD}}$ reduces ROM generation time from 6.3h to 2.5h. Figs. 1 and 2 illustrate the dependence of the iteration count on the dimension A of the projection space V_A. Fig. 3 presents the magnitude of the reflected wave versus frequency for a corner and a central radiator, respectively, as well as the directive gain at $f = 10.36$ GHz and $f = 12$ GHz.

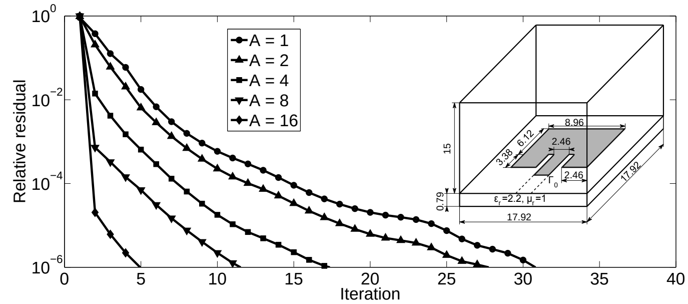


Figure 1: Relative residual of GMRES(30) method versus iteration count, using $\mathbf{P}_{\text{AD}}$. Parameter: ROM dimension A.

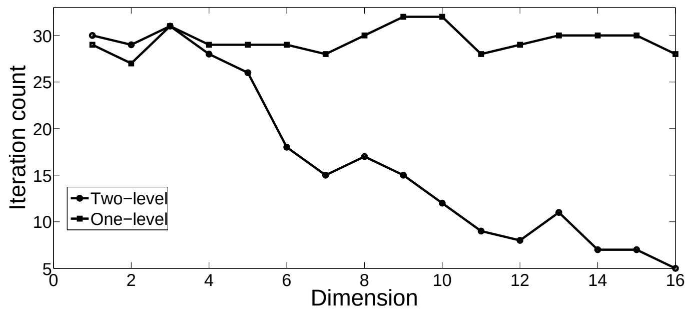
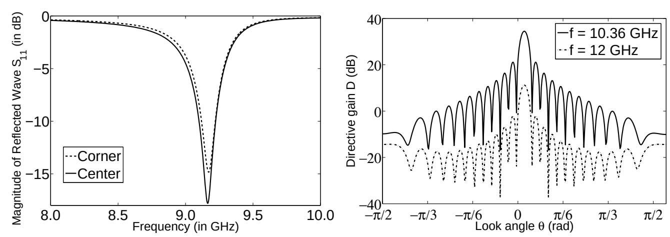


Figure 2: Iteration count versus ROM dimension V_A for the standard one-level and the proposed two-level preconditioners.



(a) Magn. of reflected wave versus frequency. Parameter: antenna location.

(b) Directive gain for principal plane $\phi = \frac{\pi}{2}$. Parameter: frequency.

Figure 3: Antenna parameters obtained from ROM.

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